

A nilpotent quotient algorithm for L -presented groups

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Abstract

The main part of this paper contains a description of a nilpotent quotient algorithm for L -presented groups and a report on applications of its implementation in the computer algebra system GAP. The appendix introduces two new infinite series of L -presented groups. Apart from being of interest in their own right, these new L -presented groups serve as examples for applications of the nilpotent quotient algorithm.

1 Introduction

A very useful group theoretic construction is that of an ascending HNN-extension: for example, this can have the form $\overline{G} = \langle G, t \mid g^t = \varphi(g) \text{ for all } g \in G \rangle$ where φ is an injective endomorphism of G . Such a construction is often used to embed a non-finitely-presented group G into a finitely presented group \overline{G} to obtain a compact description of the group G .

L -presentations have been introduced by Bartholdi [1] based on work of Lysenok [16]. They provide a framework to describe the base group G of an HNN-extension, even though it is not finitely presented, and they extend this key idea further. More precisely, an L -presentation is an expression of the form $\langle S \mid Q \mid \Phi \mid R \rangle$, where Q and R are subsets of the free group F_S and Φ is a set of endomorphisms of F_S , and it defines the group

$$F_S / \langle Q \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^{F_S},$$

where Φ^* is the monoid generated by Φ . In [1] it is proved that a broad class of groups acting on rooted trees admit an explicitly constructible finite L -presentation. Well-known examples of finitely L -presented, but not finitely presented groups are the Grigorchuk group and the Gupta-Sidki group which both play a role in the study of the famous Burnside problems.

The central aim of the main part of this paper is to describe a nilpotent quotient algorithm for finitely L -presented groups. This takes as input a finitely L -presented group G and a positive integer n and determines a nilpotent presentation for the class- n quotient

$G/\gamma_{n+1}(G)$. Thus this algorithm can also determine the abelian invariants of G and the largest nilpotent quotient of G if it exists. The algorithm described here generalises the nilpotent quotient algorithm for finitely presented groups by Nickel [18].

An implementation of the algorithm described here is available in the NQL package [14] of the computer algebra system GAP [23]. Sample applications and a report on runtimes are included below.

In the appendix to this paper, two new infinite series of L -presented groups are described. The first series generalises the Fabrykowski-Gupta group [9], the second series generalises the Gupta-Sidki group [13].

We investigated some of the groups in these series using the nilpotent quotient algorithm. As a result, we can conjecture that the generalised Fabrykowski-Gupta groups have finite width and we can identify a subfamily of these groups with width 2 and very slim lower central series. The generalised Gupta-Sidki groups do not exhibit a similarly obvious pattern in their lower central series factors and they could have infinite width.

2 More about L -presentations

In this section we recall the basic notions used to work with L -presented groups and we exhibit some examples. First we note that an L -presentation $\langle S \mid Q \mid \Phi \mid R \rangle$ is called

- finite, if S, Q, R , and Φ are finite,
- ascending, if Q is empty, and
- invariant, if $K = \langle Q \cup \bigcup_{\varphi \in \Phi} \varphi(R) \rangle^{Fs}$ satisfies $\varphi(K) \subseteq K$ for every $\varphi \in \Phi$.

In the remainder of this paper we are concerned with finite L -presentations only. Invariant L -presentations will play an important role for our algorithm. We record some basic observations on L -presentations in the following remark.

1 Remark:

- *Every ascending L -presentation is invariant. Conversely, if the L -presentation $\langle S \mid Q \mid \Phi \mid R \rangle$ is invariant, then it determines the same group as the ascending L -presentation $\langle S \mid \emptyset \mid \Phi \mid Q \cup R \rangle$.*
- *Every finite presentation $\langle S \mid R \rangle$ can be written as a finite L -presentation in the form $\langle S \mid R \mid \emptyset \mid \emptyset \rangle$ or in the form $\langle S \mid \emptyset \mid \{id\} \mid R \rangle$. The second form shows that every finite presentation can be written as a finite ascending L -presentation.*

Many of the well-known examples of L -presentations are invariant or even ascending. A famous example for this case is the Grigorchuk group, see [16] and [1] for details.

2 Example: *The Grigorchuk group can be defined by the following ascending (and hence invariant) finite L -presentation.*

$$\langle a, c, d \mid \emptyset \mid \{\sigma'\} \mid \{a^2, [d, d^a], [d, d^{acaca}]\} \rangle,$$

with

$$\sigma': \left\{ \begin{array}{lll} a & \mapsto & c^a \\ c & \mapsto & cd \\ d & \mapsto & c \end{array} \right\}.$$

We note that there are other finite L -presentations for the Grigorchuk group known. An example is the following non-ascending, but invariant L -presentation.

$$\langle a, b, c, d \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \{[d, d^a], [d, d^{acaca}]\} \rangle,$$

with

$$\sigma: \left\{ \begin{array}{lcl} a & \mapsto & c^a \\ b & \mapsto & d \\ c & \mapsto & b \\ d & \mapsto & c \end{array} \right\}.$$

3 Polycyclic and nilpotent presentations

Every finitely generated nilpotent group is polycyclic and hence can be described by a consistent polycyclic presentation. This type of presentation allows effective computations with the considered group and thus it facilitates detailed investigations of the underlying group.

In this section we recall the definitions and some of the basic ideas on polycyclic presentations with particular emphasis on finitely generated nilpotent groups. Further information and references can be found in [15], Chapter X.

A *polycyclic presentation* is a presentation on a sequence of generators, g_1, \dots, g_n say, whose relations have the following form for certain $r_1, \dots, r_n \in \mathbb{N} \cup \{\infty\}$:

$$\begin{aligned} g_i^{g_j} &= g_{j+1}^{e_{i,j,j+1}} \cdots g_n^{e_{i,j,n}} \text{ for } j < i, \\ g_i^{g_j^{-1}} &= g_{j+1}^{f_{i,j,j+1}} \cdots g_n^{f_{i,j,n}} \text{ for } j < i, \text{ and} \\ g_i^{r_i} &= g_{i+1}^{l_{i,i,i+1}} \cdots g_n^{l_{i,i,n}} \text{ for all } i \text{ with } r_i < \infty. \end{aligned}$$

Let G be the group defined by the above presentation and let $G_i = \langle g_i, \dots, g_n \rangle \leq G$. Then the above relations imply that the series $G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_n \supseteq G_{n+1} = \{1\}$ is a subnormal series with cyclic factors. We say that this is the *polycyclic series* defined by the presentation.

The factors of this polycyclic series satisfy $[G_i : G_{i+1}] \leq r_i$ for $1 \leq i \leq n$. The polycyclic presentation is called *consistent* if $[G_i : G_{i+1}] = r_i$ for $1 \leq i \leq n$. The consistency of a polycyclic presentation can be checked effectively, see [22], page 424.

Nilpotent presentations are a special case of polycyclic presentations for finitely generated nilpotent groups. Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$ denote the lower central series of G . Then we say that a polycyclic presentation of G is a *nilpotent presentation* if its polycyclic series refines the lower central series of G .

A nilpotent presentation is called *weighted*, if there exists a function $w : \{g_1, \dots, g_n\} \rightarrow \mathbb{N}$ such that $w(g_k) = 1$ if and only if $g_k \notin \gamma_2(G)$, and if $w(g_k) > 1$, then there exists a relation $g_i^{g_j} = g_i g_k$ with $j < i < k$ so that $w(g_j) = 1$ and $w(g_i) = w(g_k) - 1$.

4 Computing abelian invariants

Let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by a finite L -presentation. In this section we describe a method to determine the abelian invariants of G and a corresponding consistent nilpotent presentation of the abelian group $G/\gamma_2(G) = G/G'$.

Our method is a direct generalisation of the well-known approach to determine the abelian invariants of a finitely presented group. We refer to [22] or [15] for further information.

Let $S = \{s_1, \dots, s_m\}$ and F the free group on S . Then every element $w \in F$ is a word in $S \cup S^{-1}$, say $w = s_{i_1}^{e_1} \dots s_{i_l}^{e_l}$ with $e_i = \pm 1$. Define $a_j = \sum_{i_k=j} e_k \in \mathbb{Z}$ for $1 \leq j \leq m$ and let $\bar{w} = s_1^{a_1} \dots s_m^{a_m}$. Then \bar{w} can be considered as the collected word corresponding to w . It satisfies $wF' = \bar{w}F'$ and hence \bar{w} is a representative of the coset wF' . Translating to additive notation, we can represent \bar{w} by the vector $a_w = (a_1, \dots, a_m) \in \mathbb{Z}^m$.

Every endomorphism φ of F satisfies $\varphi(F') \subseteq F'$ and hence induces an endomorphism $\bar{\varphi}$ of F/F' . Translating to additive notation as above, we can represent $\bar{\varphi}$ by a matrix $M_\varphi \in M_m(\mathbb{Z})$ which acts by multiplication from the right on \mathbb{Z}^m as φ acts on F/F' . Thus we obtain a homomorphism $\text{End}(F) \rightarrow M_m(\mathbb{Z}) : \varphi \mapsto M_\varphi$. These constructions yield the following description of $G/\gamma_2(G)$.

3 Lemma: $G/\gamma_2(G) \cong \mathbb{Z}^m/U_G$ where $U_G = \langle a_q, a_r M_\varphi \mid q \in Q, r \in R, \varphi \in \Phi^* \rangle$.

If a subgroup V of \mathbb{Z}^m is given by a finite set of generators, then algorithms for membership testing in V and for computing the abelian invariants of the quotient \mathbb{Z}^m/V together with a corresponding minimal generating set for this quotient are described in [22], Chapter 8. Both methods rely mainly on Hermite normal form computations of matrices. The latter allows to read off a consistent nilpotent presentation for \mathbb{Z}^m/V .

To apply these methods in our setting, it remains to determine a finite generating set for the subgroup U_G of \mathbb{Z}^m as defined in Lemma 3. The following straightforward method achieves this aim. Note that this method terminates, since ascending chains of subgroups in \mathbb{Z}^m terminate.

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FiniteGeneratingSet(  $U_G$  )
  initialise  $U := \{a_q, a_r \mid q \in Q, r \in R\}$ 
  initialise  $T := \{a_r \mid r \in R\}$ 
  while  $T \neq \emptyset$  do
    choose  $t \in T$  and delete  $t$  from  $T$ 
    for  $\varphi$  in  $\Phi$  do
      compute  $s := tM_\varphi$ 
      if  $s \notin \langle U \rangle$  then add  $s$  to  $U$  and add  $s$  to  $T$ 
    end for
  end while
  return  $U$ 

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This completes our algorithm to determine the abelian invariants of G and a consistent nilpotent presentation of $G/\gamma_2(G)$ for the group G which is given by a finite L -presentation. Further, this presentation can be considered as weighted by assigning the weight 1 to every generator.

5 Computing nilpotent quotients I

The algorithm of Section 4 generalises readily to a method for determining nilpotent quotients. This is straightforward to describe, but the resulting algorithm is usually not very effective in its applications. We include a description of this generalisation here for completeness and we refer to Section 6 for a significantly more effective approach towards computing nilpotent quotients.

Let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by a finite L -presentation and let $n \in \mathbb{N}$. We wish to determine a consistent polycyclic presentation for the quotient $G/\gamma_n(G)$. As above, let F be the free group on S . Then a consistent polycyclic presentation for a group H with $H \cong F/\gamma_n(F)$ together with the corresponding natural epimorphism $\epsilon : F \rightarrow H$ can be determined using a nilpotent quotient algorithm for finitely presented groups or theoretical background on free groups.

As $\gamma_n(F)$ is invariant under every endomorphism φ of F , we obtain that φ induces an endomorphism $\bar{\varphi}$ of the quotient $F/\gamma_n(F)$. This endomorphism $\bar{\varphi}$ can be translated to an endomorphism $\tilde{\varphi}$ of H via ϵ . Thus we obtain a homomorphism $\text{End}(F) \rightarrow \text{End}(H) : \varphi \mapsto \tilde{\varphi}$. This setting yields the following description of $G/\gamma_n(G)$.

4 Lemma: $G/\gamma_n(G) \cong H/(U_G)^H$ where $U_G = \langle \epsilon(q), \tilde{\varphi}(\epsilon(r)) \mid q \in Q, r \in R, \varphi \in \Phi^* \rangle$.

Let V be a subgroup of H given by a finite set of generators. Then standard methods for polycyclically presented groups facilitate an effective membership test in V , the computation of the normal closure of V and the determination of a consistent polycyclic presentation of H/V^H . We refer to [15], Chapter X, for background.

Hence it remains to determine a finite generating set for the subgroup U_G of H as described in Lemma 4 to complete our construction for $G/\gamma_n(G)$. However, as ascending chains of subgroups in polycyclic groups terminate, we can use the same method as in Section 4 to achieve this aim.

The main deficiency of this method is that it needs to compute a consistent polycyclic presentation for the quotient $F/\gamma_n(F)$ and this quotient can easily be very large, even if the desired quotient $G/\gamma_n(G)$ is rather small.

6 Computing nilpotent quotients II

Let G be defined by a finite L -presentation and let $n \in \mathbb{N}$. In this section we describe a method to determine a consistent polycyclic presentation for $G/\gamma_n(G)$. First, in Section 6.1, we consider the special case that G is given by an invariant L -presentation. Then, in Section 6.2, we apply the special case method to obtain a method for the general case.

6.1 Invariant finite L -presentations

Let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by a finite invariant L -presentation and let $n \in \mathbb{N}$. We wish to determine a consistent nilpotent presentation for $G/\gamma_n(G)$. Note that the case $n = 1$ is trivial and the case $n = 2$ is covered by Section 4. Hence we assume that $n \geq 3$ in the following.

Our overall idea generalises the method for finitely presented groups described by Nickel [18]. Thus our basic approach is an induction on n . In the induction step, we assume that we have given a consistent weighted nilpotent presentation for $G/\gamma_{n-1}(G)$ and we seek to extend this to $G/\gamma_n(G)$. We discuss this step in more detail in the following.

First, we introduce some more notation. As before, let F be the free group on $S = \{s_1, \dots, s_m\}$ and let $K = \langle Q \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$ so that $G = F/K$. Define $K_n := K\gamma_n(F)$ for $n \in \mathbb{N}$. Then it follows that

$$G/\gamma_n(G) \cong F/K_n \text{ for all } n \in \mathbb{N}.$$

As input for the induction step we use a *nilpotent quotient system* for F/K_{n-1} as described in [18]. We briefly recall the main features of such a system as follows:

- a) a consistent weighted nilpotent presentation E/T defining a group H and having the generators e_1, \dots, e_l , say,
- b) a homomorphism $\tau : F \rightarrow H$ with kernel K_{n-1} which is defined by the images $\tau(s_i) = w_i(e_1, \dots, e_l)$ for $1 \leq i \leq m$, and
- c) for every e_j with $w(e_j) = 1$ an index $i(j)$ such that the word $w_{i(j)}(e_1, \dots, e_l)$ is of the form $w_{i(j)} = u_{i(j)}e_j$ where $u_{i(j)}$ is a word in e_1, \dots, e_{j-1} .

The definition of a weighted nilpotent presentation incorporates that every generator of weight greater than 1 in H can be written as a word in the generators of weight 1. Thus H is generated by elements of weight 1. Condition c) implies that for every generator of weight 1 we can compute a preimage in F . This yields that the homomorphism τ is surjective and it follows that

$$H \cong F/K_{n-1}.$$

The induction step now proceeds in two stages. First, we determine a nilpotent quotient system for $F/[K_{n-1}, F]$ by extending the given nilpotent quotient system. An effective method for this purpose is described in [18], Section 4. This yields

- a) a consistent weighted nilpotent presentation E^*/T^* defining a group H^* and having the generators $e_1, \dots, e_l, e_{l+1}, \dots, e_{l+d}$, say,
- b) a homomorphism $\tau^* : F \rightarrow H^*$ with kernel $[K_{n-1}, F]$ which is defined by images of the form $\tau^*(s_i) = w_i(e_1, \dots, e_l)v_i(e_{l+1}, \dots, e_{l+d})$ for $1 \leq i \leq m$, and
- c) for every e_j with $w(e_j) = 1$ we have that $w_{i(j)}(e_1, \dots, e_l)v_{i(j)}(e_{l+1}, \dots, e_{l+d}) = u_{i(j)}e_j$ as above.

Note that $K_{n-1}/[K_{n-1}, F]$ is a central subgroup in $F/[K_{n-1}, F]$. It corresponds via τ^* to the subgroup $M = \langle e_{l+1}, \dots, e_{l+d} \rangle$ of the group H^* so that H^* is a central extension of M by H .

As a second stage in the induction step of our algorithm, it now remains to determine a nilpotent quotient system for F/K_n from the given system for $F/[K_{n-1}, F]$. For this purpose we note that

$$K_n = K\gamma_n(F) = K[K, F][\gamma_{n-1}(F), F] = K[K_{n-1}, F].$$

Thus it follows that

$$F/K_n \cong H^*/\tau^*(K)$$

and it remains to determine a finite generating set for $\tau^*(K)$ as subgroup of the nilpotent group H^* . Once such a finite generating set is given, we can then use standard methods for computing with polycyclically presented groups to determine a consistent weighted nilpotent presentation for $H^*/\tau^*(K)$ and to modify the nilpotent quotient system for $F/[K_{n-1}, F]$ to such a system for the quotient $F/K_n \cong H^*/\tau^*(K)$.

We investigate $\tau^*(K)$ in more detail in the following. Recall that $M = \langle e_{l+1}, \dots, e_{l+d} \rangle$ is an abelian subgroup of H^* .

5 Lemma: $\tau^*(K) \leq M$.

Proof: This follows directly, as $K \leq \ker(\tau)$ and τ^* extends τ . •

Note that M is a finitely generated abelian group by construction. It now remains to determine a finite generating set for $\tau^*(K)$ as a subgroup of M .

6 Lemma: Every endomorphism $\varphi \in \Phi^*$ induces an endomorphism $\bar{\varphi} \in \text{End}(M)$ via τ^* and we obtain a homomorphism $\Phi^* \rightarrow \text{End}(M) : \varphi \mapsto \bar{\varphi}$.

Proof: Let $\varphi \in \Phi^*$. As the given L -presentation is invariant, it follows that $\varphi(K) \subseteq K$ holds. Clearly also $\gamma_i(F)$ is invariant under φ for every $i \in \mathbb{N}$. Thus we obtain that $K_{n-1} = K\gamma_{n-1}(F)$ and also $[K_{n-1}, F]$ are invariant under φ . Thus φ induces an endomorphism of $K_{n-1}/[K_{n-1}, F]$ and hence, via τ^* , also of M . •

This implies the following.

7 Lemma: $\tau^*(K) = \langle \tau^*(q), \bar{\varphi}(\tau^*(r)) \mid q \in Q, r \in R, \varphi \in \Phi^* \rangle$.

Proof: This follows directly by translating the defining generating set of K to generators of $\tau^*(K) \leq M$. •

As M is finitely generated abelian, it satisfies the ascending chain condition. Thus a finite generating set for $\tau^*(K)$ can be computed from the description given in Lemma 7 using a similar approach to the algorithm ‘FiniteGeneratingSet’ of Section 4.

We summarise our resulting algorithm for the induction step as follows. Let $\mathcal{Q}(F/L)$ denote the nilpotent quotient system for a quotient F/L of F .

InductionStep($\mathcal{Q}(F/K_{n-1})$)

 Compute a nilpotent quotient system $\mathcal{Q}(F/[K_{n-1}, F])$ (see [18]).

 Induce every $\varphi \in \Phi$ to $\bar{\varphi} \in \text{End}(M)$.

 Induce every $g \in Q \cup R$ to $\tau^*(g) \in M$.

 Determine a finite generating set for $\tau^*(K)$ using Lemma 7 and ‘FiniteGeneratingSet’.

 Determine a consistent weighted nilpotent presentation for $H^*/\tau^*(K)$.

 Return $\mathcal{Q}(F/K_n)$ as modification of $\mathcal{Q}(F/[K_{n-1}, F])$.

6.2 Arbitrary finite L -presentations

Now let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by an arbitrary finite L -presentation and let $n \in \mathbb{N}$. We wish to determine a consistent polycyclic presentation for $G/\gamma_n(G)$. As above, let F be the free group on S and denote $K = \langle Q \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$. Our method proceeds in the following 3 steps.

Step 1: We determine an invariant finite L -presentation $\langle S \mid \overline{Q} \mid \Phi \mid R \rangle$ defining a group \overline{G} , say, so that its kernel $\overline{K} = \langle \overline{Q} \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$ satisfies $\overline{K} \subseteq K$.

Step 2: We determine the nilpotent quotient of the larger group \overline{G} as $H := \overline{G}/\gamma_n(\overline{G})$ using the method of Section 6.1.

Step 3: We determine the finite set U of images of $Q \setminus \overline{Q}$ in H and obtain $G/\gamma_n(G) \cong H/\langle U \rangle^H$ using standard methods for polycyclically presented groups.

Step 1 requires some further explanation. First note that we could always choose $\overline{Q} = \emptyset$ and thus obtain a fully automatic algorithm. However, the effectivity of the above method relies critically on finding an L -presentation in Step 1 that yields a possibly “small” subgroup $\langle U \rangle^H$. (“Small” means here that the difference in the numbers of generators of the polycyclic presentation for H and its induced presentation for $H/\langle U \rangle^H$ is small.) Thus it may be of interest to supply a “nice” L -presentation for Step 1 by other means. However, there is no general algorithm for finding such a “nice” L -presentation available at current.

7 Sample applications and runtimes

The algorithm described in this paper has been implemented in the GAP package NQL [14]. In this section we outline runtimes for some sample applications of this algorithm and thus exhibit the scope and the range of possible applications of our algorithm.

All timings displayed below have been obtained on an Intel Pentium 4 computer with clock speed 2.80 GHz by applying the NQL algorithm with a time limit of two hours. Then the computation has been stopped and the resulting nilpotent quotient together with the total time used to obtain this quotient has been listed.

7.1 Some well-known groups

There are various interesting examples of finitely L -presented, but not finitely presented groups known. We list some of them in the following; the Fabrykowski-Gupta group and the Gupta-Sidki group are treated in detail in the next sections.

- G : the Grigorchuk group with its L -presentation in [16].
- \tilde{G} : the Grigorchuk supergroup with its L -presentation in [1], Theorem 4.6.
- BSV : the Brunner-Sidki-Vieira group [8] with its L -presentation in [1], Theorem 4.4.
- Δ : the Basilica group [11] with its L -presentation in [6].
- B : the Baumslag group [7] with its L -presentation in [1], Theorem 4.2.
- L : the Lamplighter group with its L -presentation in [1], Theorem 4.1.

Table 1 briefly describes the application of our algorithm to these groups. It lists whether the considered groups have ascending or non-invariant L -presentations, it briefly describes the obtained nilpotent quotients by their classes and the number of generators in their nilpotent presentations and it exhibits the runtimes used to determine the nilpotent quotients.

Table 1 shows that our algorithm has a significantly better performance on ascending L -presentations than on non-invariant ones. In the case of a non-invariant L -presentation, the column ‘gens’ of Table 1 lists in brackets the number of generators of the invariant L -presentation used in Step 1 of the method in Section 6.2.

Group	prop	class	gens	time (h:min)
G	asc	80	130	1:53
\tilde{G}	asc	47	127	1:56
BSV	asc	34	171	1:27
Δ	asc	39	220	1:47
B	non-inv	11	12 (423)	0:21
L	non-inv	9	10 (253)	0:04

Table 1: Some well-known groups

In the remainder of this subsection, we outline and discuss the lower central series quotients $\gamma_i(*)/\gamma_{i+1}(*)$ for the groups in Table 1 in more detail. To shorten notation, we outline lists in collected form; that is, if an entry a in a list appears in n consecutive places, then we write $a^{[n]}$ instead of n times a .

The lower central series quotients of the Grigorchuk group G are known by theoretical results of Rozhkov [20], see also [10]. Our computations confirm the following theorem.

8 Theorem: (See [20]) *The Grigorchuk group G satisfies that*

$$rk(\gamma_i(G)/\gamma_{i+1}(G)) = \left\{ \begin{array}{ll} 3 \text{ or } 2 & \text{if } i = 1 \text{ or } 2 \text{ resp.} \\ 2 & \text{if } i \in \{2 \cdot 2^k + 1, \dots, 3 \cdot 2^k\} \\ 1 & \text{if } i \in \{3 \cdot 2^k + 1, \dots, 4 \cdot 2^k\} \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Grigorchuk supergroup \tilde{G} we computed $\gamma_i(\tilde{G})/\gamma_{i+1}(\tilde{G})$ for $1 \leq i \leq 64$. The resulting groups are elementary abelian 2-groups with ranks

$$4, 3^{[2]}, 2, 3^{[2]}, 2^{[2]}, 3^{[4]}, 2^{[4]}, 3^{[8]}, 2^{[8]}, 3^{[16]}, 2^{[16]}.$$

This induces the following conjecture.

9 Conjecture: *The Grigorchuk supergroup \tilde{G} satisfies that*

$$rk(\gamma_i(\tilde{G})/\gamma_{i+1}(\tilde{G})) = \left\{ \begin{array}{ll} 3 & \text{if } i \in \{2 \cdot 2^k + 1, \dots, 3 \cdot 2^k\} \\ 2 & \text{if } i \in \{3 \cdot 2^k + 1, \dots, 4 \cdot 2^k\} \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Brunner-Sidki-Vieira group BSV the Jennings series is completely determined in [2]. But there have been only the first 4 quotients of its lower central series known so far. We computed $\gamma_i(BSV)/\gamma_{i+1}(BSV)$ for $1 \leq i \leq 43$ and obtained the following abelian invariants:

$$\begin{aligned} &(0, 0), (0), (8), \\ &(8), (4, 8), (2, 8), \\ &(2, 2, 8)^{[2]}, (2, 2, 4, 8)^{[2]}, (2, 2, 2, 8)^{[2]}, \\ &(2, 2, 2, 2, 8)^{[4]}, (2, 2, 2, 2, 4, 8)^{[4]}, (2, 2, 2, 2, 2, 8)^{[4]}, \\ &(2, 2, 2, 2, 2, 2, 8)^{[8]}, (2, 2, 2, 2, 2, 2, 4, 8)^{[8]}, (2, 2, 2, 2, 2, 2, 2, 8)^{[3]} \end{aligned}$$

This induces the following conjecture, where $I(*)$ denotes the abelian invariants of a group.

10 Conjecture: *The Brunner-Sidki-Vieira group BSV satisfies that*

$$I(\gamma_i(BSV)/\gamma_{i+1}(BSV)) = \left\{ \begin{array}{ll} (2^{[2k]}, 8) & \text{if } i \in \{3 \cdot 2^k + 1, \dots, 4 \cdot 2^k\} \\ (2^{[2k]}, 4, 8) & \text{if } i \in \{4 \cdot 2^k + 1, \dots, 5 \cdot 2^k\} \\ (2^{[2k+1]}, 8) & \text{if } i \in \{5 \cdot 2^k + 1, \dots, 6 \cdot 2^k\} \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Basilica group Δ we computed $\gamma_i(\Delta)/\gamma_{i+1}(\Delta)$ for $1 \leq i \leq 48$ and obtained the following abelian invariants:

$$\begin{aligned} &(0, 0), (0), (4)^{[2]}, (4, 4), (2, 4) \\ &(2, 2, 4)^{[2]}, (2, 2, 2, 4), (2, 2, 2, 2, 4)^{[2]}, (2, 2, 2, 4), \\ &(2, 2, 2, 2, 4)^{[4]}, (2, 2, 2, 2, 2, 4)^{[2]}, (2, 2, 2, 2, 2, 2, 4)^{[4]}, (2, 2, 2, 2, 2, 2, 4)^{[2]}, \\ &(2, 2, 2, 2, 2, 2, 2, 4)^{[8]}, (2, 2, 2, 2, 2, 2, 2, 2, 4)^{[4]}, (2, 2, 2, 2, 2, 2, 2, 2, 2, 4)^{[8]}, (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4)^{[4]}, \end{aligned}$$

This induces the following conjecture.

11 Conjecture: *The Basilica group Δ satisfies that*

$$I(\gamma_i(\Delta)/\gamma_{i+1}(\Delta)) = \left\{ \begin{array}{ll} (2^{[2k+2]}, 4) & \text{if } i \in \{6 \cdot 2^k + 1, \dots, 8 \cdot 2^k\} \\ (2^{[2k+3]}, 4) & \text{if } i \in \{8 \cdot 2^k + 1, \dots, 9 \cdot 2^k\} \\ (2^{[2k+4]}, 4) & \text{if } i \in \{9 \cdot 2^k + 1, \dots, 11 \cdot 2^k\} \\ (2^{[2k+3]}, 4) & \text{if } i \in \{11 \cdot 2^k + 1, \dots, 12 \cdot 2^k\} \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

Baumslag's group B and the Lamplighter group L are both known to be metabelian. This yields that their lower central series patterns can be deduced theoretically. We include the abelian invariants of $\gamma_i(*)/\gamma_{i+1}(*)$ for these two groups as far as we computed them for completeness:

$$\text{for } B: (3, 0), 3^{[10]} \quad \text{for } L: (2, 0), 2^{[8]}.$$

7.2 The Fabrykowski-Gupta group and its generalisation

An infinite series of groups with an ascending finite L -presentation are introduced in Appendix 8.1: the *generalised Fabrykowski-Gupta groups* Γ_p with $p \geq 3$. We used these groups as sample inputs for the nilpotent quotient algorithm; the results are outlined in this section. They support that this family of groups contains some very interesting groups.

First, we briefly summarise the results of our algorithm on Γ_p for some small p in Table 2 using the same format as in Table 1. Note that all considered L -presentations are ascending in this case. Additionally, the table contains a column noting whether our algorithm found a maximal nilpotent quotient.

Group	max quot	class	gens	time (h:min)
Γ_3	no	71	112	1:50
Γ_4	no	66	146	1:55
Γ_5	no	53	60	1:58
Γ_6	yes	3	4	0:00
Γ_7	no	44	50	1:37
Γ_8	no	52	116	1:47
Γ_9	no	58	84	1:54
Γ_{10}	yes	5	6	0:00
Γ_{11}	no	33	35	1:48
Γ_{12}	yes	6	7	0:00
Γ_{14}	yes	7	8	0:00
Γ_{15}	yes	5	6	0:00
Γ_{18}	yes	15	16	0:06
Γ_{20}	yes	6	7	0:02
Γ_{21}	yes	7	8	0:04

Table 2: Fabrykowski-Gupta groups Γ_p for some small p

In the following we discuss the lower central series factors of the groups Γ_p in more detail. First, we consider the case that p is not a prime-power. We summarise our results in the following conjecture.

12 Conjecture: *If p is not a prime-power, then Γ_p has a maximal nilpotent quotient.*

Next, we consider the case that p is a prime. For the smallest possible prime $p = 3$, there is a theoretical description of the lower central series factors of Γ_3 known from [3]. Our computations confirm the following theorem.

13 Theorem: (See [3])

$$rk(\gamma_i(\Gamma_3)/\Gamma_{i+1}(\Gamma_3)) = \left\{ \begin{array}{ll} 2 \text{ or } 1 & \text{if } i = 1 \text{ or } 2 \text{ resp.} \\ 2 & \text{if } i \in \{3^k + 2, \dots, 2 \cdot 3^k + 1\} \\ 1 & \text{if } i \in \{2 \cdot 3^k + 2, \dots, 3^{k+1} + 1\} \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the primes $p = 5, 7, 11$, we list the lower central series factors $\gamma_i(\Gamma_p)/\gamma_{i+1}(\Gamma_p)$ obtained by our algorithm in the following. Note that all determined factors are elementary abelian p -groups and we list their ranks in collected form.

- Γ_5 : $2, 1^{[3]}, 2, 1^{[13]}, 2^{[5]}, 1^{[30]}$.
- Γ_7 : $2, 1^{[5]}, 2, 1^{[33]}, 2^{[4]}$.
- Γ_{11} : $2, 1^{[9]}, 2, 1^{[22]}$.

Thus if p is a prime, then the groups Γ_p seem to have a very slim lower central series. It seems very likely that these groups exhibit a lower central series pattern similar to that of Γ_3 and it would be very interesting to spot and prove this. However, for this purpose a larger computed sequence would be helpful. We only formulate the following conjecture.

14 Conjecture: *Let p be an odd prime. Then Γ_p is a group of width 2.*

Finally, we consider the case that p is a prime-power, say q^n . All the obtained lower central series factors $\gamma_i(\Gamma_p)/\gamma_{i+1}(\Gamma_p)$ are q -groups in this case and, except for some initial entries, they are elementary abelian. Again, it would be interesting to find and prove a general pattern for these factors.

- Γ_4 : $(4, 4), (4), 2^{[4]}, 3^{[3]}, 2^{[13]}, 3^{[12]}, 2^{[32]}$.
- Γ_8 : $(8, 8), (8), (4)^{[4]}, 2, 1, 2^{[2]}, 3, 2, 3^{[2]}, 4, 3^{[8]}, 2^{[23]}, 3^{[5]}, 2$.
- Γ_9 : $(9, 9), (9)^{[2]}, 1^{[5]}, 2^{[6]}, 3, 2^{[17]}, 1^{[26]}$.

Thus the groups Γ_p still seem to be of finite width, but the width grows with the exponent n in the power $p = q^n$.

7.3 The Gupta-Sidki group and its generalisations

The Gupta-Sidki group GS has originally been introduced in [13] and has become famous for its role in connection with the Burnside problems. In Appendix 8.2 generalisations GS_p of this group for all odd primes p are introduced and finite non-invariant L -presentation for these groups are obtained. In this section, we investigate the groups GS_p using our nilpotent quotient algorithm.

As a preliminary step, we discuss two different strategies to determine nilpotent quotients of GS_p with our algorithm. First, we can apply our algorithm to the non-invariant L -presentation of GS_p outlined in Appendix 8.2. This is straightforward, but usually yields only very limited results, as our algorithm is not effective on non-invariant L -presentations. For a second, more effective approach we use the structure of GS_p as exhibited in Appendix 8.2. Every GS_p is of the form $GS_p \cong D_p \rtimes C_p$, where D_p is generated by $\{\sigma_1, \dots, \sigma_p\}$ and the cyclic group C_p acts by permuting these elements cyclically. An ascending L -presentation for D_p is also included in Appendix 8.2. Now we can apply our algorithm to the ascending L -presentation of D_p and determine $D_p/\gamma_c(D_p)$ for some c . Then, defining $H_p = D_p/\gamma_c(D_p) \rtimes C_p$, we obtain $GS_p/\gamma_i(GS_p) \cong H_p/\gamma_i(H_p)$ for all $i \leq c$.

Table 3 summarises runtimes and a brief overview on the results of our algorithm applied to GS_p for $p = 3, 5, 7$. The table uses the same notation as Table 1. Instead of a column ‘prop’ it has a column ‘strategy’ which lists the used strategy and hence also determines

whether our algorithm was applied to an ascending or non-invariant L -presentation. Note that we applied the nilpotent quotient algorithm for 2 hours in all cases. Thus the runtimes for GS_3 with strategy 1 show that the first 5 quotients are fast to obtain, while the 6th quotients takes over 2 hours and hence did not complete. Further, Table 3 shows that strategy 2 is more successful on GS_3 than strategy 1; a feature that we also observed for other GS_p .

Group	strategy	class	gens	time (h:min)
GS_3	1	5	8 (215)	0:02
GS_3	2	25	51	1:44
GS_5	2	9	22	1:09
GS_7	2	6	13	0:59

Table 3: The Gupta-Sidki groups GS_p for some small primes p

Next, we discuss the obtained results for the lower central series of Γ_p and H_p in more detail. Our computational results for GS_3 agree with the following theoretical description of $\gamma_i(GS_3)/\gamma_{i+1}(GS_3)$ from [3].

15 Theorem: (See [3])

Let $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2}$ for $n \geq 3$. Then, for $n \geq 2$, the rank of $\gamma_n(GS_3)/\gamma_{n+1}(GS_3)$ is the number of ways of writing $n - 1$ as a sum $k_1\alpha_1 + \dots + k_t\alpha_t$ with all $k_i \in \{0, 1, 2\}$.

For all primes $p > 3$ no theoretical description of the lower central series factors of GS_p is available; it would be very interesting to obtain one. In the following we outline our computed results for the ranks of $\gamma_i(H_p)/\gamma_{i+1}(H_p)$. These are isomorphic to $\gamma_i(GS_p)/\gamma_{i+1}(GS_p)$ for all $i \leq c$, where c is the class listed in Table 3, and they are epimorphic images otherwise. This is indicated by a bar | in the list below.

- H_5 : $2, 1, 2^{[2]}, 3, 2, 3^{[2]}, 4 \mid 4^{[3]}, 3^{[3]}, 4^{[4]}, 3, 4^{[2]}, 6^{[3]}, 5, 4, 2^{[3]}, 1^{[3]}$.
- H_7 : $2, 1, 2^{[2]}, 3^{[2]}, 4 \mid 3, 4^{[2]}, 5^{[6]}, 4, 3^{[5]}, 2^{[3]}, 1^{[2]}$.

7.4 Some finitely presented groups

Each finitely presented group $\langle X \mid R \rangle$ has a finite ascending L -presentation of the form $\langle X \mid \emptyset \mid \{\text{id}\} \mid R \rangle$ and hence the algorithm described here also applies to finitely presented groups. The following finitely presented groups are from [18].

$$\begin{aligned}
G_1 &= \text{free group on 3 generators} \\
G_2 &= \text{free group on 4 generators} \\
G_3 &= \langle a, b \mid [a, [a, [a, b]]], [b, [b, [a, b]]] \rangle \\
G_4 &= \langle x, y \mid [[y, x], y], [[[[[y, x], x], x], x], x] \rangle.
\end{aligned}$$

Runtimes for these groups are outlined Table 4.

Group	class	gens	time
G_1	8	1318	0:11
G_2	6	964	0:04
G_3	17	272	1:19
G_4	20	275	1:31

Table 4: Some finitely presented groups

Comparing these runtimes with the runtimes of the nilpotent quotient algorithm of the NQ package [19] shows that the latter is significantly faster. This is mainly due to the fact that the NQL package [14] is implemented in GAP code and uses the available machinery for computing with polycyclic groups in GAP, while the NQ package [19] is implemented in C code and all underlying machinery has been designed for computing with nilpotent groups.

8 Appendix (by Laurent Bartholdi)

One of the main reasons to introduce L -presentations was the desire to understand better some examples of “self-similar” groups, and in particular striking patterns along their lower central series.

By a self-similar group we mean a group G acting on the set of words X^* over an alphabet X , and preserving the length and ‘prefix’ relation on X^* . This means that for every $x \in X, g \in G$ there are $y \in X, h \in G$ with

$$g(xw) = yh(w) \text{ for all } w \in X^*. \quad (1)$$

These groups have appeared across a wide range of mathematics, answering classical questions, for example on torsion and growth, in infinite group theory as well as establishing new links with complex dynamics. See the monograph [17] for more details.

We will capture the data in equation (1) defining a self-similar group as follows: $(y, h) = \Psi(g, x)$, for some function $\Psi : G \times X \rightarrow X \times G$. It suffices to specify Ψ on $S \times X$ for some generating set S of G .

In [1, 8] a few ‘sporadic’ calculations of L -presentations for self-similar groups are described. A systematic construction of L -presentations for all self-similar groups arising as the iterated monodromy group of a quadratic, complex polynomial is given in [5].

In this appendix, I describe two more infinite families of L -presentations. These naturally generalize the groups constructed by Gupta and Sidki [13] and Fabrykowski and Gupta [9]. The former is an elementary family of infinite, finitely generated, torsion p -groups, while the latter is a group of intermediate word-growth. The ‘generalized’ groups that I consider here are a small variations of their constructions.

I will present the calculations in compact form, mainly relying on Theorem 3.1 in [1]. They generalize the case $p = 3$ described in that paper, correcting at the same time typographical and/or copying errors. The presentations given in [1] for the Gupta-Sidki

and Fabrykowski-Gupta groups are not correct as is. The presentations given in [21] and [12] are in principle equivalent, but not as readily amenable to manipulation and understanding.

The obtained L -presentations may be accessed in the forthcoming GAP package FR, see [4].

8.1 Generalized Fabrykowski-Gupta groups

For a fixed integer $p \geq 3$ let $X = \mathbb{Z}/p\mathbb{Z} = \{0, \dots, p-1\}$ and consider the group $G = \langle a, r \rangle$ acting on X^* via

$$\Psi(a, x) = (x + 1, 1), \quad \Psi(r, 0) = (0, r), \quad \Psi(r, 1) = (1, a), \quad \Psi(r, x) = (x, 1) \text{ else.}$$

In the case $p = 3$, this is the Fabrykowski-Gupta group, which was shown in [9] to be of “subexponential word-growth”. The following theorem states the main result of this section.

16 Theorem: *For any p , the generalized Fabrykowski-Gupta group G admits a finite ascending L -presentation with generators α, ρ . With $\sigma_i = \rho^{\alpha^i}$ for $1 \leq i \leq p$, its iterated relations are*

$$\mathcal{R} = \left\{ \alpha^p, \left[\sigma_i^{\sigma_{i-1}^n}, \sigma_j^{\sigma_{j-1}^m} \right], \sigma_i^{-\sigma_{i-1}^{n+1}} \sigma_i^{\sigma_{i-1}^n \sigma_{i-1}^{\sigma_{i-1}^m}} \mid \begin{array}{l} 1 \leq i, j \leq p \\ 2 \leq |i - j| \leq p - 2 \\ 0 \leq m, n \leq p - 1 \end{array} \right\},$$

and its only endomorphism is defined by $\varphi(\alpha) = \rho^{\alpha^{-1}}$ and $\varphi(\rho) = \rho$. (Note that some relators in \mathcal{R} are redundant, since the elements $\sigma_1, \dots, \sigma_p$ are conjugate. For example, one may fix $i = 1$.)

The proof of this theorem follows the strategy of [1], Theorem 3.1, which proceeds as follows. We first consider the finitely presented group

$$\Gamma = \langle \alpha, \rho \mid \alpha^p, \rho^p \rangle$$

mapping naturally onto G by ‘greek \mapsto latin’. We then consider the subgroup $\Delta = \langle \rho^{\alpha^i} : 0 \leq i < p \rangle$ of Γ , and the homomorphism $\Phi : \Delta \rightarrow \Gamma^p$, defined by

$$\Phi(\rho^{\alpha^i}) = (1, \dots, \rho, \alpha, \dots, 1) \text{ with the } \rho \text{ at position } i.$$

We compute a presentation of $\Phi(\Delta)$; the kernel of Φ is generated by the set \mathcal{R} of Φ -preimages of relators in that presentation. Finally, we seek a section $\Sigma : \Gamma \rightarrow \Delta$ of the projection of Φ on its first coordinate. We then have, for all $x \in \Gamma$,

$$\Phi(\Sigma(x)) = (x, ?, \dots, ?),$$

where the $?$ stand for unimportant elements of Γ . The following result now allows to read off a finite ascending L -presentation for G .

17 Scholium: An L -presentation of G is given by generators α, ρ ; endomorphism Σ ; and iterated relations \mathcal{R} .

In the remainder of this section, we apply this strategy to determine a finite L -presentation for G and thus prove Theorem 16. A presentation of $\Phi(\Delta)$ can be determined by the Reidemeister-Schreier method. Consider first the presentation

$$\Pi = \langle \alpha_1, \dots, \alpha_p, \rho_1, \dots, \rho_p | \alpha_i^p, \rho_i^p, [\alpha_i, \alpha_j], [\alpha_i, \rho_j], [\rho_i, \rho_j] \text{ for } i \neq j \rangle;$$

this is a presentation of Γ^p , and $\Phi(\Delta)$ is the subgroup $\langle \sigma_i := \rho_i \alpha_{i+1} \rangle$. Here and below indices are all treated modulo p . We rewrite this presentation as

$$\Pi = \langle \alpha_1, \dots, \alpha_p, \sigma_1, \dots, \sigma_p | \alpha_i^p, \sigma_i^p, [\alpha_i, \alpha_j], [\alpha_i, \sigma_j], [\sigma_i \alpha_{i+1}^{-1}, \sigma_j \alpha_{j+1}^{-1}] \text{ for } i \neq j \rangle.$$

Next we rewrite the last set of relations either as $[\sigma_i, \sigma_j]$, if $2 \leq |i - j| \leq p - 2$, or as $\sigma_i^{\alpha_i} = \sigma_i^{\sigma_{i-1}}$, in the other cases.

We choose as Schreier transversal all p^p elements $\alpha_1^{n_1} \dots \alpha_p^{n_p}$. The Schreier generating set easily reduces to $\{\sigma_{i,n} := \sigma_i^{\alpha_i^{n_i}}\}$. The Schreier relations are all $[\sigma_{i,n}, \sigma_{j,m}]$ for $2 \leq |i - j| \leq p - 2$, and all $\sigma_{i,n+1} = \sigma_{i,n}^{\sigma_{i-1,m}}$.

In particular, we can use this last relation (with $m = 0$) to eliminate all generators $\sigma_{i,n}$ with $n \neq 0$, replacing them by $\sigma_i^{\sigma_{i-1}^n}$. We obtain $\Phi(\Delta) = \langle \sigma_1, \dots, \sigma_p | \sigma_1^p, \dots, \sigma_p^p, \mathcal{R} \rangle$, with

$$\mathcal{R} = \left\{ \left[\sigma_i^{\sigma_{i-1}^n}, \sigma_j^{\sigma_{j-1}^m} \right] \text{ whenever } 2 \leq |i - j| \leq p - 2, \quad \sigma_i^{-\sigma_{i-1}^{n+1}} \sigma_i^{\sigma_{i-1}^n} \sigma_{i-1}^{\sigma_{i-2}^m} \right\}.$$

Note that Σ satisfies $\Phi(\Sigma(x)) = (x, \alpha^?, \dots, \alpha^?)$ for all $x \in \Gamma$, and thus clearly induces a monomorphism of G . Hence we obtain the L -presentation of Theorem 16 for G .

8.2 Generalized Gupta-Sidki groups

Assume now that p is an odd prime, and consider the following group $G = \langle a, t \rangle$: its action on X^* is specified by

$$\Psi(a, x) = (x + 1, 1), \quad \Psi(t, 0) = (0, t), \quad \Psi(r, x) = (x, a^x) \text{ if } x > 0.$$

If $p = 3$, this is the original Gupta-Sidki group, which was shown in [13] to be an infinite, finitely generated, 3-torsion group.

With a similar notation as above, we consider $\Gamma = \langle \alpha, \tau | \alpha^p, \tau^p \rangle$, the normal closure Δ of τ , and the map $\Phi : \Delta \rightarrow \Gamma^p$ defined by

$$\Phi(\tau^{\alpha^i}) = (\dots, \alpha^{p-1}, \tau, \alpha, \alpha^2, \dots) \text{ with the } \tau \text{ at position } i.$$

In the group

$$\Gamma^p = \Pi = \langle \alpha_1, \dots, \alpha_p, \tau_1, \dots, \tau_p | \alpha_i^p, \tau_i^p, [\alpha_i, \alpha_j], [\alpha_i, \tau_j], [\tau_i, \tau_j] \text{ for } i \neq j \rangle,$$

we consider now the subgroup $\Phi(\Delta) = \langle \sigma_i := \tau_i \alpha_{i+1} \dots \alpha_{i+k}^k \dots \alpha_{i-1}^{-1} \rangle$. We rewrite the presentation of Π as

$$\Pi = \langle \alpha_1, \dots, \alpha_p, \sigma_1, \dots, \sigma_p \mid \alpha_i^p, \sigma_i^p, [\alpha_i, \alpha_j], [\alpha_i, \sigma_j], [\sigma_i \alpha_j^{j-i}, \sigma_j \alpha_i^{i-j}] \text{ for } i \neq j \rangle.$$

We choose as Schreier transversal all p^p elements $\alpha_1^{n_1} \dots \alpha_p^{n_p}$. The Schreier generating set easily reduces to $\{\sigma_{i,n} := \sigma_i^{\alpha_i^{n_i}}\}$. The Schreier relations become $\sigma_{i,m+i}^{-1} \sigma_{j,n+i}^{-1} \sigma_{i,m+j} \sigma_{j,n+j}$. Furthermore, an easy calculation gives

$$\left[\sigma_i^{(j-k)e} \sigma_j^{(k-i)e}, \sigma_k^{(i-j)e} \sigma_i^{(j-k)e} \right] = \sigma_{i,(j-i)(i-k)e}^{-2(j-k)e} \sigma_i^{2(j-k)e}. \quad (2)$$

For all $\ell > 0$, we may choose arbitrarily j, k such that i, j, k are all distinct and $(j-i)(i-k)/2(j-k) \equiv \ell \pmod{p}$, and use equation (3) to express $\sigma_{i,\ell}$ in terms of $\sigma_i, \sigma_j, \sigma_k$, namely

$$\sigma_{i,\ell} = \sigma_i \left[\sigma_i^{1/2} \sigma_j^{(k-i)/2(j-k)}, \sigma_k^{(i-j)/2(j-k)} \sigma_i^{1/2} \right]^{-1}. \quad (3)$$

Finally, we may also use equation (3) to construct an endomorphism Σ ; we summarize:

18 Theorem: *The subgroup $D = \langle t \rangle^G$ of the Gupta-Sidki p -group admits a finite ascending L -presentation with generators $\sigma_1, \dots, \sigma_p$ generating a free group Δ ; iterated relations*

$$\mathcal{R} = \left\{ \sigma_i^p; \sigma_{i,m+i}^{-1} \sigma_{j,n+i}^{-1} \sigma_{i,m+j} \sigma_{j,n+j} \right\};$$

and an endomorphism $\Sigma : \Delta \rightarrow \Delta$, defined by

$$\Sigma(\sigma_i) = \sigma_{1,i} \text{ as given in equation (3).}$$

It is not possible to extend Σ to an endomorphism of Γ . However, the extension of a finitely L -presented group by a finite group is again L -presented; in the present case, it is a simple matter, from the L -presentation of D , to construct the split extension $G = D \rtimes_{\zeta} \mathbb{Z}/p\mathbb{Z}$, in which the automorphism ζ of D cyclically permutes the generators.

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